MATH2048 Honored Linear Algebra II

Midterm Examination 1

Please show all your steps, unless otherwise stated. Answer all **five** questions.

1. (**15pts**) Let $\beta = \{1, x, x^2, \cdots, x^n\}$ be the standard ordered basis for $P_n(\mathbb{R})$, $\beta' = \{1, 1+x, \cdots, 1+x^n\}$ be another ordered basis for $P_n(\mathbb{R})$; and let $\gamma = \{e_1, e_2, \cdots, e_{n+1}\}$ be the standard ordered basis for F^{n+1}, $\gamma' = \{e_1, e_1 + e_2, e_2 + e_3, \cdots, e_n + e_{n+1}\}$ be another ordered basis for F^{n+1}. Consider the following linear transformation

$$\begin{aligned} T : P_n(\mathbb{R}) &\to F^{n+1} \\ p(x) &\mapsto [(x \cdot p(x))']_\beta \end{aligned}$$

   (a) Find $[T]_\beta^\gamma$.

   (b) Find A, the change of coordinate matrix from β' to β; and find B, the change of coordinate matrix from γ' to γ.

   (c) Show that $[T]_{\beta'}^{\gamma'} = B^{-1}[T]_\beta^\gamma A$.

2. (**15pts**) Let $p_0(x) = x$, consider the following mapping

$$\begin{aligned} T : P_2(\mathbb{R}) &\to \mathbb{R}^4 \\ p(x) &\mapsto ((p_0 \cdot p)(-1),\ (p_0 \cdot p)(1),\ (p_0 \cdot p)'(0),\ (p_0 \cdot p)'(1)) \end{aligned}$$

   for any $p \in P(\mathbb{R})$ and p' refers to the first derivative of p.

   (a) Show that T is a linear transformation.

   (b) Find $[T]_\beta^\gamma$, where β and γ are the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^4$ respectively.

   (c) Use dimension theorem to determine whether T is one-to-one. Please explain your answer with details.

3. (**20pts**) Consider the following mapping

$$\begin{aligned} T : P_n(\mathbb{R}) &\to \mathbb{R}^{n+1} \\ p(x) &\mapsto \left(\int_0^1 p(t)dt, \int_1^2 p(t)dt, \cdots, \int_n^{n+1} p(t)dt \right) \end{aligned}$$

   (a) Show that T is a linear transformation.

   (b) Show that T is an isomorphism. (Hint: Any non-zero polynomial of degree n has at most n distinct zeros)

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4. (**25pts**) Let V be a finite-dimensional vector space over the field F and $U \subset V$ be a subspace of V. Define U_0 as follows:

$$U_0 = \{g \in V^* : g(\mathbf{u}) = 0 \text{ for all } \mathbf{u} \in U\}.$$

   (a) Let $\beta = \{\mathbf{u}_1, ..., \mathbf{u}_m\}$ be a basis of U. Extend it to a basis $\gamma = \{\mathbf{u}_1, ..., \mathbf{u}_m, \mathbf{u}_{m+1}, ..., \mathbf{u}_n\}$ of V. Suppose $\alpha = \{\varphi_1, ..., \varphi_n\}$ is the dual basis of γ. Prove that $\{\varphi_{m+1}, ..., \varphi_n\}$ is a basis of U_0. Deduce that $\dim(U_0) = \dim(V) - \dim(U)$.

   (b) **(A little bit challenging)** Consider the inclusion map $I : U \to V$ defined by $I(\mathbf{u}) = \mathbf{u}$ for all $\mathbf{u} \in U$.

      i. Let $I^* : V^* \to U^*$ be the dual map of I. Show that $N(I^*) = U_0$ and $R(I^*) = U^*$

      ii. Using the dimension theorem, prove that $\dim(U_0) = \dim(V) - \dim(U)$.

5. (**25pts**) Suppose V_1, V_2, V_3, V_4 and V_5 are vector spaces over the same field F. For each $i = 1, 2, 3, 4$, $T_i : V_i \to V_{i+1}$ is a linear transformation between V_i and V_{i+1}. The relationship can be described by the following diagram.

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \xrightarrow{T_3} V_4 \xrightarrow{T_4} V_5$$

The above diagram is said to be *exact* if $N(T_{j+1}) = R(T_j)$ for $j = 1, 2$ and 3.

   (a) Suppose V_1, V_4 and V_5 are all zero spaces (that is, a vector space consisting only of the zero vector). Prove that the diagram is exact if and only if T_2 is an isomorphism between V_2 and V_3.

   (b) **(Challenging)** Suppose V_1, V_2, V_3, V_4 and V_5 are all finite-dimensional. Prove that if

$$V_1 \xrightarrow{T_1} V_2 \xrightarrow{T_2} V_3 \xrightarrow{T_3} V_4 \xrightarrow{T_4} V_5$$

   is exact, then

$$V_5^* \xrightarrow{T_4^*} V_4^* \xrightarrow{T_3^*} V_3^* \xrightarrow{T_2^*} V_2^* \xrightarrow{T_1^*} V_1^*$$

   is also exact.

END OF PAPER